Threshold feedback control for a collective flashing ratchet: Threshold dependence

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(Received 11 May 2006; revised manuscript received 27 July 2006; published 9 October 2006)

We study the threshold control protocol for a collective flashing ratchet. In particular, we analyze the dependence of the current on the values of the thresholds. We have found analytical expressions for the small threshold dependence both for the few and for the many particle cases. For few particles the current is a decreasing function of the thresholds, thus, the maximum current is reached for zero thresholds. In contrast, for many particles the optimal thresholds have a nonzero finite value. We have numerically checked the relation that allows to obtain the optimal thresholds for an infinite number of particles from the optimal period of the periodic protocol. These optimal thresholds for an infinite number of particles give good results for many particles. In addition, they also give good results for few particles due to the smooth dependence of the current up to these threshold values.

DOI: 10.1103/PhysRevE.74.041109 PACS number(s): 05.40.−a, 02.30.Yy

I. INTRODUCTION

Ratchets or Brownian motors are rectifiers of thermal fluctuations. This rectification is usually achieved through the introduction of an external deterministic or stochastic perturbation in a system that is or becomes asymmetric under spatial inversion [1]. Over the last years ratchets have been studied due to their theoretical and experimental relevance. From a practical point of view the ratchet effect has many potential applications in biology, condensed matter and nanotechnology [1,2].

Ratchets can be viewed as controllers that act on stochastic systems with the aim of inducing directed motion through the rectification of the fluctuations. In particular, flashing ratchets are thermal fluctuation rectifiers based on switching on and off a periodic potential [3,4]. Several studies deal with the problem of optimizing the particle current [5] or the efficiency [6] in these systems. However, they all consider only open-loop controllers (as that obtained with a periodic or random switching). Recently, feedback control protocols have been introduced in the context of collective ratchets [7,8]. In the feedback control protocols the action of the controller depends on the state of the system. This feedback control, or closed-loop control, can be implemented in systems where the state of the system is monitored (as occurs in some experimental setups with colloidal particles [9]).

In this paper we study one of these closed-loop controls, the *threshold control*, previously introduced in Ref. [8]. The structure of the paper is as follows. In the next section we present the mathematical model of the collective flashing ratchet with the threshold control protocol and we discuss briefly other protocols that have been studied in recent papers. Later, in Sec. III, we analyze the dependence of the average center-of-mass velocity on the thresholds, obtaining analytical approximated expressions that are compared with the numerical results. In Sec. III A we study the small thresholds dependence (distinguishing the many particles case and the few particles case), while in Sec. III B we discuss the dependence of the average center-of-mass velocity for any thresholds and any number of particles. Finally, we present our conclusions in Sec. IV.

II. THE MODEL

We consider N Brownian particles with positions $x_i(t)$ at temperature T within a ratchet potential $V(x)$, and whose dynamics is described by the overdamped Langevin equations

$$\gamma\dot{x}_i(t) = \alpha(t)F(x_i(t)) + \xi_i(t), \quad i = 1, \ldots, N, \tag{1}$$

with γ the friction coefficient (related to the diffusion coefficient D through Einstein's relation $D=k_BT/\gamma$) and $\xi_i(t)$ Gaussian white noises of zero mean satisfying the fluctuation-dissipation relation $\langle\xi_i(t)\xi_j(t')\rangle=2\gamma k_BT\delta_{ij}\delta(t-t')$. The force is given by $F(x)=-V'(x)$ and α is a control parameter that can take only two possible values, $\alpha=0$ (potential off) or $\alpha=1$ (potential on).

Several control strategies have been studied in order to maximize the particle current in this system. The *optimal periodic switching* [7,8] consists on switching the potential on during a time period $\mathcal{T}_{\mathrm{on}}$ and switching it off during $\mathcal{T}_{\mathrm{off}}$. Note that it is an open-loop control protocol and therefore the results are independent of N. This protocol is the periodic

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flashing ratchet, that has been widely studied both theoretically and experimentally [1,2]. The *maximization of the center-of-mass instant velocity* protocol has been introduced and studied in Ref. [7]. It consist of switching the potential on only if the net force would be positive. Therefore, it is a closed-loop control protocol, because it needs information about the state of the system in order to operate. This is the best strategy for a single particle. However, for a large number of particles the system gets trapped with the potential on or off and then the average steady state current tends to zero as N increases [7]. Another closed-loop control protocol, the *threshold control*, was proposed in Ref. [8] to avoid this effect. In this paper we analyze it further.

The net force per particle is

$$f(t) = \frac{1}{N}\sum_{i=1}^{N} F(x_i(t)). \tag{2}$$

On the other hand, given the state of the system $x_i(t)$, a good estimator for the time derivative of $f(t)$ can be obtained using Langevin equation (1) and Ito calculus (see Ref. [8]),

$$\dot{f}_{\rm exp} \equiv \frac{1}{\gamma N}\sum_i \alpha(t)F(x_i(t))F'(x_i(t)) + \frac{k_BT}{\gamma N}\sum_i F''(x_i(t)). \tag{3}$$

The *maximization of center-of-mass instant velocity* protocol has $\alpha(t)=\Theta(f(t))$, with Θ the Heaviside function [$\Theta(x)=1$ if $x>0$, else $\Theta(x)=0$]. In contrast, the *threshold control* policy has two thresholds $u_{\rm on}\geqslant 0$ and $u_{\rm off}\leqslant 0$ which induce earlier switchings that permit to avoid the trapping. When $f(t)$ decreases below $u_{\rm on}$ we switch off the potential, although the net force is still positive, in order to avoid the trapping. Analogously the potential is switched on if the net force per particle increases above $u_{\rm off}$, so we induce the flipping of the system before $f(t)$ is positive. Therefore, the *threshold control* is given by

$$\alpha(t) = \begin{cases} 1 & \text{if } f(t) \geqslant u_{\rm on}, \\ 1 & \text{if } \quad u_{\rm off} < f(t) < u_{\rm on} \quad \text{and } \dot{f}_{\rm exp}(t) \geqslant 0, \\ 0 & \text{if } \quad u_{\rm off} < f(t) < u_{\rm on} \quad \text{and } \dot{f}_{\rm exp}(t) < 0, \\ 0 & \text{if } f(t) \leqslant u_{\rm off}. \end{cases} \tag{4}$$

This scheme removes the long decaying tails in the evolution of the net force preventing the trapping. Note that this protocol and the maximization of the center-of-mass instant velocity protocol are feedback controls or closed-loop controls. The threshold control protocol in the zero thresholds limit gives the maximization of the center-of-mass instant velocity protocol.

III. THRESHOLD CONTROL STRATEGY

A. Small thresholds

In this section we analyze the threshold control strategy improving and extending the analytic expressions found for the maximization of the center-of-mass instant velocity protocol [7].

1. Many particles: quasideterministic approximation

For many particles (large N) the net force has a quasideterministic behavior. It can be described in terms of two contributions, a deterministic contribution f^{∞} (given by the behavior for an infinite number of particles) plus a small stochastic contribution

$$f(t) = f^{\infty}(t) + \text{fluctuations}. \tag{5}$$

This approximate description has proven to be fruitful in order to understand the behavior of these ratchets in the many particle case [7].

The deterministic contribution, that reflects the behavior of the system for an infinite number of particles ($N\to\infty$), can be described through a particle distribution $\rho(x,t)$ that evolves according to the mean-field Fokker-Planck equation $\gamma\partial_t\rho(x,t)=[-\alpha(t)\partial_x F(x)+k_BT\partial_x^2]\rho(x,t)$. The net force per particle is a deterministic function $f^{\infty}(t)=\langle F(x)\rangle_\rho \equiv \int_0^L dx\rho(x,t)F(x)$, with L the period of the ratchet potential. The net force is zero for the equilibrium distribution when the potential is on and also when it is off. We denote by $f_\nu^{\infty}(t)$ with ν=on, off the value of the deterministic part of the net force when the system has been evolving with the potential on or off, respectively, a time t after a switching. After a certain time τ_ν it can be approximately described by [7]

$$f_\nu^{\infty}(t) = C_\nu e^{-\lambda_\nu(t-\tau_\nu)}. \tag{6}$$

C_ν and λ_ν are constants that are obtained by fitting the net force obtained with the Fokker-Planck equation. In order to obtain $f_{\rm on}^{\infty}(t)$ we evolve the equilibrium distribution for the off potential with the Fokker-Planck equation with the potential on, i.e., we assume that the system was close to the equilibrium state for the off potential before the off-on switching. We proceed analogously for $f_{\rm off}^{\infty}(t)$.

On the other hand, the amplitude of the fluctuations of the net force f can be estimated as [7]

$$\Sigma = \sqrt{\langle f^2(t)\rangle - \langle f(t)\rangle^2} \simeq \sqrt{\frac{\langle F^2\rangle_\rho - \langle F\rangle_\rho^2}{N}} \sim \frac{V_0}{L\sqrt{a(1-a)N}}. \tag{7}$$

This simple result is a good estimation of the amplitude of the fluctuations for potentials with characteristic height V_0 and asymmetry a. For example, the potential

$$V(x) = \frac{2V_0}{3\sqrt{3}}\left[\sin\left(\frac{2\pi x}{L}\right) + \frac{1}{2}\sin\left(\frac{4\pi x}{L}\right)\right], \tag{8}$$

that we have used for the figures of this paper, has characteristic height V_0 and characteristic asymmetry $a=1/3$ (where aL is defined as the minimum distance between a minimum and a maximum of the potential, with L being the period of the potential).

We have already provided estimations for both the deterministic part of the net force per particle and the amplitude of its fluctuations. This will allow us to calculate the average current.

First, we compute the characteristic times during which the potential remains on, $t_{\rm on}$, and off, $t_{\rm off}$. In the threshold control protocol the switching happens when the force crosses the threshold value with the appropriate slope [see Eq. (4)]. When the threshold is crossed the equality $u_\nu = f_\nu(t_\nu)$ is satisfied, with $f_\nu(t)$ the value of the net force a time t after a switching. Therefore, using the quasideterministic approximation (5) we obtain for the characteristic times

$$|f_\nu^\infty(t_\nu)| - \Sigma = |u_\nu|. \tag{9}$$

Using Eq. (6) we get the following explicit equations for the characteristic times:

$$t_\nu = \tau_\nu + \frac{1}{\lambda_\nu} \ln \frac{|C_\nu|}{|u_\nu| + \Sigma}, \tag{10}$$

with Σ given by Eq. (7). Moreover, Eq. (6) implies that this approximation is valid for $t_\nu \gtrsim \tau_\nu$, where τ_ν are the transient times for each dynamics [afterwards, Eq. (6) is a good approximation]. This implies $|u_\nu| + \Sigma \ll |C_\nu|$, that can be expressed as $|u_\nu| + \Sigma \ll \max_t |f_\nu^\infty(t)|$ by using $|C_\nu| \sim \max_t |f_\nu^\infty(t)|$. As $\Sigma \sim 1/\sqrt{N}$, we see that this approximation is valid for small thresholds and large number of particles.

We now compute the average displacement of the center-of-mass during an on-off period. Note that the center-of-mass moves only when the potential is on, because when it is off the dynamics is purely diffusive. Therefore, as the center-of-mass position is $x_{\text{c.m.}} = \Sigma_i x_i / N$, its average displacement during an on-off cycle in the many particle case is given by using the evolution equations (1) as

$$\Delta x_{\text{c.m.}}(t_{\text{on}}) = \frac{1}{\gamma} \int_0^{t_{\text{on}}} f_{\text{on}}^\infty(t) dt. \tag{11}$$

The integration of the late time expression (6) with $\nu = \text{on}$ suggests a functional form

$$\Delta x_{\text{c.m.}}(t_{\text{on}}) = \Delta x_{\text{on}} (1 - e^{-t_{\text{on}}/\Delta t_{\text{on}}}). \tag{12}$$

This functional form fits well the function $\Delta x_{\text{c.m.}}(t_{\text{on}})$ obtained from the numerical integration of the Fokker-Planck equation, and this fit is used to determine Δx_{on} and Δt_{on}. We have seen that the inclusion of the characteristic time Δt_{on} improves the analytical results obtained in Ref. [7] [there it was assumed $\Delta x_{\text{c.m.}}(t_{\text{on}}) = \Delta x_{\text{on}}$]. This better estimation of the average displacement improves the results for the intermediate regime of not-so-large number of particles. Furthermore, the whole expression (12) is also necessary to improve the results for nonzero thresholds. When thresholds are enlarged the frequency of switching increases and therefore the times t_{on} decrease. This implies a shorter displacement, as Eq. (12) predicts.

The previous results allow us to give an approximate expression for the average center-of-mass velocity in the stationary regime,

$$\langle \dot{x}_{\text{c.m.}} \rangle_{\text{st}} \equiv \lim_{t\to\infty} \frac{x_{\text{c.m.}}(t) - x_{\text{c.m.}}(0)}{t} = \frac{\Delta x_{\text{on}}}{t_{\text{on}} + t_{\text{off}}} (1 - e^{-t_{\text{on}}/\Delta t_{\text{on}}})$$

$$= \frac{\Delta x_{\text{on}} [1 - A (u_{\text{on}} + \Sigma)^{1/(\lambda_{\text{on}} \Delta t_{\text{on}})}]}{B - \frac{1}{\lambda_{\text{on}}} \ln(u_{\text{on}} + \Sigma) - \frac{1}{\lambda_{\text{off}}} \ln(|u_{\text{off}}| + \Sigma)}, \tag{13}$$

with Σ given by Eq. (7), and A and B given by

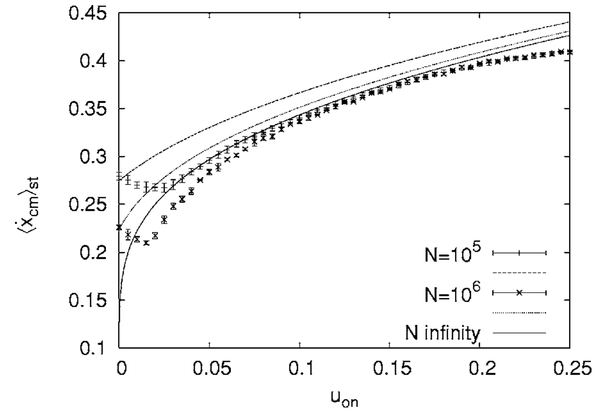

FIG. 1. Average of the center-of-mass velocity $\langle \dot{x}_{\text{c.m.}} \rangle_{\text{st}}$ as a function of the threshold u_{on} for numbers of particles $N = 10^5$, 10^6 and the limit $N \to \infty$ for the potential (8) with $V_0 = 5k_BT$. Analytical quasideterministic approximation (13) (lines) and numerical results from Langevin equations (1) (points with error bars). We have taken $u_{\text{off}} = -u_{\text{on}}$. (Units, $L=1$, $D=1$, $k_BT=1$.)

$$A = e^{-\tau_{\text{on}}/\Delta t_{\text{on}}} C_{\text{on}}^{-1/(\lambda_{\text{on}} \Delta t_{\text{on}})},$$

$$B = \tau_{\text{on}} + \tau_{\text{off}} + \frac{1}{\lambda_{\text{on}}} \ln C_{\text{on}} + \frac{1}{\lambda_{\text{off}}} \ln |C_{\text{off}}|.$$

The final expression in Eq. (13) shows the explicit dependence on the thresholds u_{on}, u_{off}, and on the amplitude of the force fluctuations Σ; all the other parameters are determined by the dynamics for an infinite number of particles with zero thresholds. Equation (13) has been obtained in the quasideterministic approximation and therefore is valid when the number of particles N is large and the thresholds are small, as discussed after Eq. (10). We have verified that it gives good estimations inside its regime of validity. In particular, for zero thresholds Eq. (13) is better than the formula obtained in Ref. [7] thanks to the introduction of the characteristic time Δt_{on}. (The formula in Ref. [7] is recovered for $u_{\text{on}} = u_{\text{off}} = 0$ and $\Delta t_{\text{on}} = 0$.)

Figures 1–3 compare the predictions of the quasideterministic approximation, Eq. (13), with the numerical results for the threshold control protocol applied with the potential (8) and $V_0 = 5k_BT$. For this potential the fit to the Fokker-Planck evolution gives $C_{\text{on}} = 0.67 k_BT/L$, $\tau_{\text{on}} = 0.058 L^2/D$, $\lambda_{\text{on}} = 28 D/L^2$, $C_{\text{off}} = -0.74 k_BT/L$, $\tau_{\text{off}} = 0.037 L^2/D$, $\lambda_{\text{off}} = 39 D/L^2$, and $\Delta x_{\text{on}} = 0.08L$, $\Delta t_{\text{on}} = 0.05 L^2/D$.

In Fig. 1 we plot the current as a function of the threshold u_{on} (with $u_{\text{off}} = -u_{\text{on}}$) comparing the quasideterministic approximation (13) and the numerical results obtained from the Langevin evolution equations (1). We see that the quasideterministic approximation gives a good estimation of the current. However, it fails to predict the minimum located at low threshold values. This minimum is caused by a secondary effect that has not been accounted in the deduction of the analytic formula. This secondary effect is due to the fact that nonzero thresholds have the disadvantage of not being instantly optimal, because they imply switching on the potential when the force is still negative and switching off the potential when the force is still positive. In addition, for very

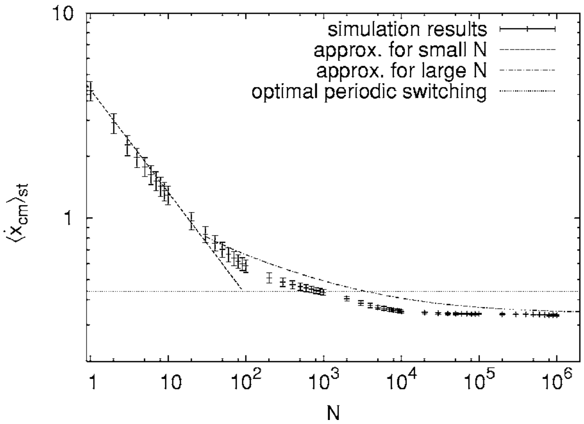

FIG. 2. Average of the center-of-mass velocity $\langle \dot{x}_{\mathrm{c.m.}} \rangle_{\mathrm{st}}$ as a function of the number of particles N for the potential (8) with $V_0 = 5k_BT$ and for thresholds $u_{\mathrm{on}}=0.1$ and $u_{\mathrm{off}}=-0.1$. The simulations results obtained solving numerically the Langevin equations (1) (points with error bars) are compared with the quasideterministic approximation for large N [Eq. (13)] and the pure stochastic approximation for small N [Eq. (19)]. The dotted horizontal straight line corresponds to the periodic switching protocol with optimal periods. (Units, $L=1$, $D=1$, $k_BT=1$.)

small thresholds the switchings are not induced much earlier than they would be with zero thresholds due to the force fluctuations. Thus, there is a minimum located at thresholds of order $1/\sqrt{N}$, the magnitude of the force fluctuations. For larger threshold this secondary effect of the thresholds is overcompensated by the main effect of avoiding the undesired trapping of the dynamics. This main effect allows to have similar average displacements of the particles in a shorter on-off cycle time. Therefore, larger thresholds increase the average center-of-mass velocity.

Figures 2 and 3 compare analytic and numerical results for the current as a function of the number of particles for fixed nonzero thresholds: Fig. 2 for $u_{\mathrm{on}}=-u_{\mathrm{off}}=0.1k_BT/L$ and Fig. 3 for $u_{\mathrm{on}}=0.6k_BT/L$ and $u_{\mathrm{off}}=-0.4k_BT/L$ (which are the optimal values for an infinite number of particles). In Fig. 2 we see that the quasideterministic approximation gives a good estimation for large number of particles. In Fig. 3 the estimate is more rough due to the fact that the thresholds do not strictly verify the validity condition of the quasideterministic approximation ($|u_\nu|+\Sigma \ll |C_\nu|$). Another interesting result we have found is that for fixed nonzero thresholds the average velocity as a function of N tends to a constant asymptotic value for large number of particles, as Eq. (13) predicts. For an infinite number of particles the force fluctuation vanishes, thus, this asymptotic value is given by Eq. (13) evaluated at $\Sigma=0$. See Figs. 2 and 3.

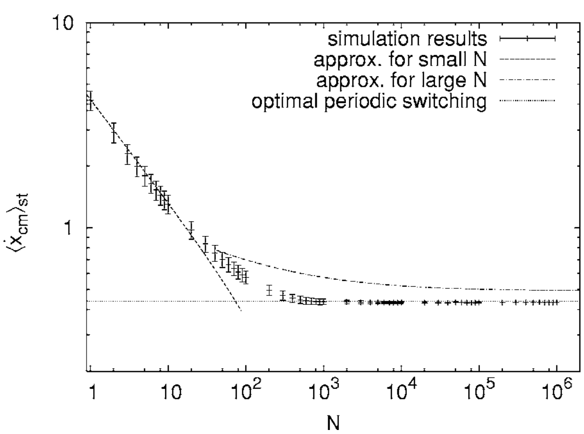

FIG. 3. Average of the center-of-mass velocity $\langle \dot{x}_{\mathrm{c.m.}} \rangle_{\mathrm{st}}$ as a function of the number of particles N for the potential (8) with $V_0 = 5k_BT$ and for thresholds $u_{\mathrm{on}}=0.6$ and $u_{\mathrm{off}}=-0.4$ (optimal values for $N\to\infty$). The simulations results obtained solving numerically the Langevin equations (1) (points with error bars) are compared with the quasideterministic approximation for large N [Eq. (13)] and the pure stochastic approximation for small N [Eq. (19)]. The dotted horizontal straight line corresponds to the periodic switching protocol with optimal periods. (Units, $L=1$, $D=1$, $k_BT=1$.)

The optimal threshold protocol gives the same current or better than the optimal periodic control [8] (Fig. 3). In particular, for an infinite number of particles the force fluctuations become negligible and the threshold control becomes equivalent to a periodic switching. The relation between the thresholds and the periods [8]

$$u_\nu = f_\nu^\infty(\mathcal{T}_\nu) \tag{14}$$

is obtained here as the limit $N\to\infty$, i.e., $\Sigma=0$, of Eq. (9). This relation permits to get the optimal thresholds for an infinite number of particles from the optimal periods just using the functions $f_{\mathrm{on}}^\infty(t)$ and $f_{\mathrm{off}}^\infty(t)$ obtained numerically from the Fokker-Planck equation. This avoids the need of integrating numerically N coupled Langevin equations for large values of N. We have numerically checked that the expression (14) gives the optimal thresholds (see Sec. III B and Fig. 7).

2. Few particles: pure stochastic approximation

When we have few particles the situation is the opposite to that considered in the preceding section and the net force has nearly a pure stochastic behavior. A binomial distribution is found for the net force probability distribution, $p(f)$, in Ref. [7] under the approximations that the position of the particles are statistically independent and that the probability of finding a particle in the interval $[0, aL]$ is a. For simplicity this binomial distribution for the net force can be approximated by a Gaussian distribution

$$p(f) \simeq \frac{1}{\sqrt{2\pi\Sigma^2}} e^{-f^2/(2\Sigma^2)}, \tag{15}$$

with Σ the amplitude of the fluctuations of the net force, that is given by Eq. (7). Neglecting the time correlations in the net force, the average center-of-mass velocity for the threshold protocol [Eq. (4)] is given by

$$\langle \dot{x}_{\mathrm{c.m.}} \rangle_{\mathrm{st}} = \frac{1}{\gamma}\int_{u_{\mathrm{on}}}^{\infty} f p(f) df + \frac{1}{\gamma}\int_{u_{\mathrm{off}}}^{u_{\mathrm{on}}} f p_+(f) df, \tag{16}$$

with $p_+(f)$ the probability of having a net force f and a nonnegative value of $\dot{f}_{\mathrm{exp}}$ $[p_+(f)\sim p(f)/2]$. This implies that, in the validity range of this small N approximation [$\Sigma \gtrsim \max_t |f^\infty(t)|$], the current is a decreasing function of the threshold u_{on}, as can be easily proven comparing the results

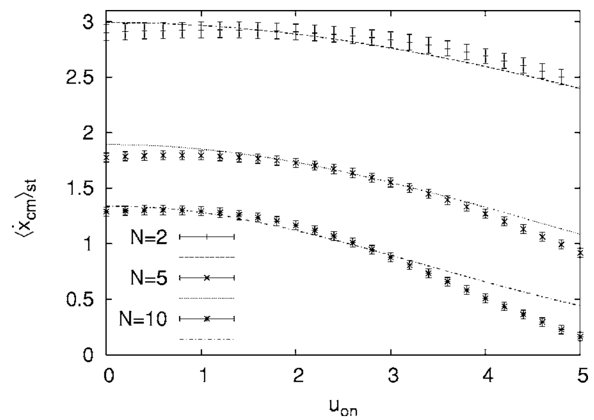

FIG. 4. Average of the center-of-mass velocity $\langle \dot{x}_{\text{c.m.}} \rangle_{\text{st}}$ as a function of the threshold u_{on} for $N=2$, 5, and 10 particles for the potential (8) with $V_0=5k_BT$. Analytical pure stochastic approximation (19) (lines) and numerical results from Langevin equations (1) (points with error bars) are compared. We have taken $u_{\text{off}}=-u_{\text{on}}$. (Units, $L=1$, $D=1$, $k_BT=1$.)

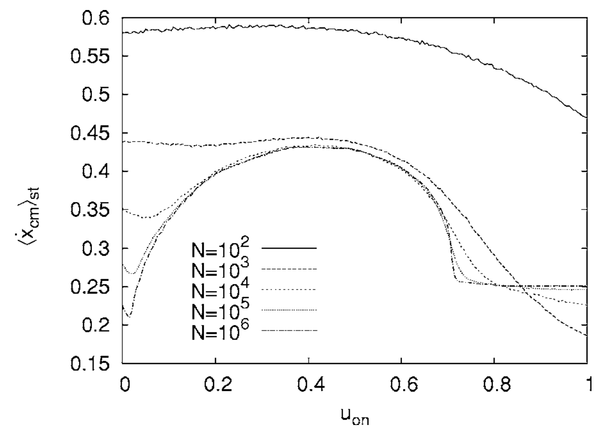

FIG. 5. Average of the center-of-mass velocity $\langle \dot{x}_{\text{c.m.}} \rangle_{\text{st}}$ as a function of the threshold u_{on} with $u_{\text{off}}=-u_{\text{on}}$ for various N. The lines correspond to the numerical solution of the Langevin equations (1) for the potential (8) with $V_0=5k_BT$. (Units, $L=1$, $D=1$, $k_BT=1$.)

for u'_{on} and u_{on} with $0 \leqslant u'_{\text{on}} < u_{\text{on}}$. Equation (16) gives

$$\langle \dot{x}_{\text{c.m.}} \rangle_{\text{st}}(u'_{\text{on}}) - \langle \dot{x}_{\text{c.m.}} \rangle_{\text{st}}(u_{\text{on}}) = \frac{1}{\gamma}\int_{u'_{\text{on}}}^{u_{\text{on}}} f p_-(f) df, \qquad (17)$$

with $p_-(f) \equiv p(f) - p_+(f) \geqslant 0$. Thus, the last term in the previous expression is non-negative implying

$$\langle \dot{x}_{\text{c.m.}} \rangle_{\text{st}}(u'_{\text{on}}) - \langle \dot{x}_{\text{c.m.}} \rangle_{\text{st}}(u_{\text{on}}) \geqslant 0. \qquad (18)$$

Analogously, it can be shown that for $0 \geqslant u'_{\text{off}} > u_{\text{off}}$ we have $\langle \dot{x}_{\text{c.m.}} \rangle_{\text{st}}(u'_{\text{off}}) - \langle \dot{x}_{\text{c.m.}} \rangle_{\text{st}}(u_{\text{off}}) = (1/\gamma)\int_{u_{\text{off}}}^{u'_{\text{off}}}(-f)p_+(f)df \geqslant 0$. This shows that the average center-of-mass velocity is a decreasing function for increasing modulus of the thresholds. Therefore, for small N we get the maximum current for zero thresholds.

For small thresholds we have found an approximate explicit analytical expression for the current. If $u_{\text{off}} \simeq -u_{\text{on}}$ the contribution of the second integral in Eq. (16) is generally small, because it is the integration of a nearly odd function in a nearly symmetric interval around zero. On the other hand, the contribution of the first integral is greater provided the thresholds are small ($u_{\text{on}} \lesssim \Sigma$). Then, neglecting the second integral we obtain

$$\langle \dot{x}_{\text{c.m.}} \rangle_{\text{st}} \simeq \frac{\Sigma}{\gamma\sqrt{2\pi}} e^{-u_{\text{on}}^2/(2\Sigma^2)}. \qquad (19)$$

(Note that for $u_{\text{on}}=0$ we recover the zero threshold result found in Ref. [7].) This expression, Eq. (19), gives good predictions when we have few particles and small thresholds. In particular, we show in Figs. 2–4 that it correctly predicts the threshold and particle number dependence of the current, even for $u_{\text{on}} \sim \Sigma \simeq 3.4$ when $N=10$ (Fig. 4).

B. General thresholds

In the preceding section we have studied the threshold protocol when the moduli of the thresholds are small, obtaining approximate analytical expressions for the current. In contrast, in this section we study the threshold protocol for general thresholds (that are in general beyond the applicability range of the previous analytical expressions). This study is done performing numerical simulations of the Langevin equation of the threshold protocol for general values of the thresholds.

1. $u_{\text{off}}=-u_{\text{on}}$

Let us discuss first the results for thresholds that are related by $u_{\text{off}}=-u_{\text{on}}$.

In the few particle case, when the thresholds are small the current decreases exponentially with the square of the threshold as we have already seen [see Eq. (19)]. However, as the rate of the exponential is small, we nearly have a plateau near the maximum at zero thresholds, as shown in Figs. 4 and 5. On the other hand, for very large thresholds Eq. (19) is no longer valid and the current decreases faster than the exponential. Note that the current continues to be a decreasing function, as predicted by Eq. (18) (valid for any threshold values in the few particle case). See Figs. 4 and 5.

In contrast, in the many particle case the maximum of the current is no longer at zero thresholds, but at a finite value. As we have explained before, the introduction of thresholds has the advantage of inducing earlier switchings. This avoids the undesired trapping that otherwise is present for large N implying low current values. The presence of thresholds allows to have similar average displacements of the particles in a shorter on-off cycle time, and therefore increases the average center-of-mass velocity. However, if the thresholds are too large the losses in the displacement become more important than the gains of shortening the on-off cycle time. Therefore, the current has a maximum located at a finite value of the thresholds in the many particle case (Fig. 5). (The tiny minimum in the small threshold region is related to another effect: the disadvantages of choosing a not instantly optimal protocol. For a more detailed explanation see Sec. III A 1.) Another important result in the many particle case is that the maximum obtained for the current as a function of the

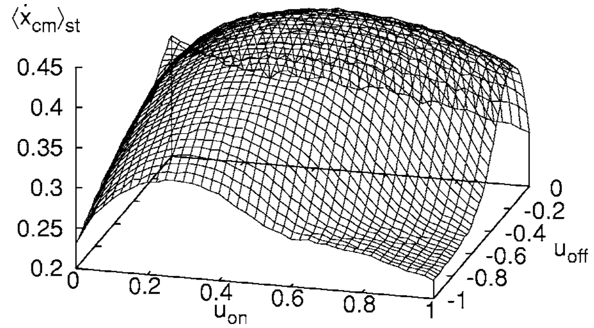

FIG. 6. Thresholds dependence of the average of the center-of-mass velocity $\langle \dot{x}_{c.m.} \rangle_{st}$ for $N=10^4$ particles in the potential (8) with $V_0=5k_BT$. The grid has been obtained integrating numerically Langevin equations (1) for different thresholds u_{on} and u_{off}. (Units, $L=1$, $D=1$, $k_BT=1$.)

threshold magnitude is quite flat and nearly independent of the number of particles. See Fig. 5.

In summary, in the many particle case the current has a maximum for nonzero thresholds whose position is nearly independent of the number of particles. On the other hand, for few particles the current is maximum for zero thresholds. However, in the few particle case the current is nearly the same up to thresholds of the order of the thresholds that give the maximum for the many particle case (see Figs. 4 and 5). This has an important implication: the optimal threshold values for the many particle case give currents close to the maximum for *any* number of particles.

2. $u_{off} \neq -u_{on}$

The study of the current for completely general thresholds u_{on} and u_{off} (without restrictions) reveals that the behavior is analogous to that described previously. In fact, the optimal thresholds for large number of particles are located not far from the line $u_{on}=-u_{off}$, and these thresholds give currents close to the maximum for any number of particles. (See Figs. 6 and 7.)

As we have already commented in the preceding section, for an infinite number of particles the force fluctuations becomes negligible and the threshold protocol becomes equivalent to a periodic switching. This implies the relation (14) between the optimal periods $\mathcal{T}_{on}$ and $\mathcal{T}_{off}$, and the optimal thresholds u_{on} and u_{off}, that we have numerically checked (see Fig. 7). Therefore, these relations permit to obtain the optimal thresholds for an infinite number of particles from the optimal periods, just using the functions $f_{on}^{\infty}(t)$ and $f_{off}^{\infty}(t)$ obtained numerically from the Fokker-Planck equation. These thresholds give good results for large number of particles. Moreover, it is important to note that these threshold values also give currents close to the maximum in the few particles case due to the smooth dependence for small thresholds (see Figs. 5–7).

In particular, we have seen that for the potential (8) with $V_0=5k_BT$ the optimal switching periods are approximately $\mathcal{T}_{on}=0.06L^2/D$ and $\mathcal{T}_{off}=0.05L^2/D$. Therefore, with just a Fokker-Planck simulation for the potential we have found that a good estimation of the optimal thresholds is given by

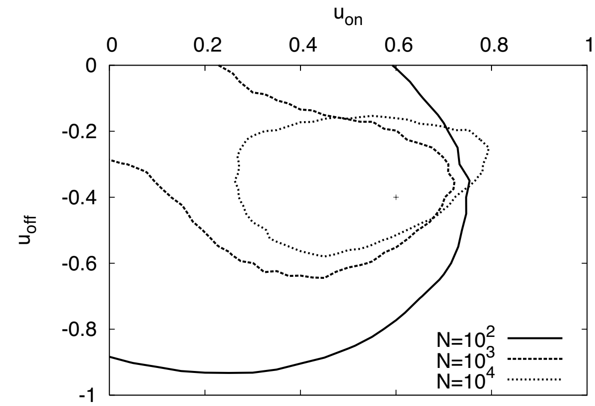

FIG. 7. Thresholds contour lines corresponding to the value of the average of the center-of-mass velocity $\langle \dot{x}_{c.m.} \rangle_{st}$ 5% below its maximum for $N=10^2$, $N=10^3$, and $N=10^4$ particles in the potential (8) with $V_0=5k_BT$. The contour line for $N=10^5$ is already very similar to that for $N=10^4$. The point corresponds to the optimal thresholds for $N\to\infty$ obtained from the optimal periods using Eq. (14). (Units, $L=1$, $D=1$, $k_BT=1$.)

$u_{on}=f_{on}^{\infty}(\mathcal{T}_{on})=0.6k_BT/L$ and $u_{off}=f_{off}^{\infty}(\mathcal{T}_{off})=-0.4k_BT/L$, in good agreement with Fig. 7.

IV. CONCLUSIONS

In this paper we have analyzed the *threshold control* protocol for a collective flashing ratchet. We have studied the threshold dependence of the current in this closed-loop control protocol. The quasideterministic (for many particles) approximation [7] has been improved through the introduction of an additional characteristic time giving better results for not-so-many particles. Both the quasideterministic and the stochastic (for few particles) approximations [7] have been applied to the threshold control protocol. This has led to analytical expressions for large and small number of particles. We have computed numerically the current dependence on the thresholds and on the number of particles obtaining a good agreement between analytical and numerical results in the validity range of our assumptions. We have also compared these results with the *optimal periodic switching* protocol.

We have seen that for many particles the current has a maximum for nonzero thresholds whose position is nearly independent of the number of particles. On the other hand, for few particles we have demonstrated that the current increases as thresholds moduli decrease, so the maximum current is reached at zero thresholds. However, the current is nearly the same up to thresholds of the order of the optimal thresholds for the many particle case. This implies that the optimal thresholds values for the many particle case give currents close to the maximum for any number of particles. The optimal thresholds for an infinite number of particles can be obtained from the optimal periods of the periodic protocol just solving the Fokker-Planck equation in two particular cases (potential on and off, see Sec. III A). Therefore, we can get a good estimation of the optimal thresholds for many particles, that also gives currents close to the optimal for any number of particles as we have shown.

The closed-loop threshold control gives the same current as the optimal protocols for the one particle case and for an infinite number of particles, and it gives high currents in between. However, obtaining the best protocol for the maximization of the current is still an open question.

In this work, and in previous ones [7,8], we have seen that, thanks to the information about the fluctuations obtained through the feedback, the performance of the system can be increased. This increase of the performance has thermodynamical limitations that have been studied in a general context for the efficiency [10]. We are now working in order to get a deeper understanding of this interplay between the information and the increase of the performance.

ACKNOWLEDGMENTS

The authors acknowledge financial support from the Ministerio de Ciencia y Tecnología (Spain) through the research projects BFM2003-02547/FISI, FIS2005-24376–E, and FIS2006-05895. In addition, one of the authors (M.F.) thanks the Universidad Complutense de Madrid (Spain) and one of the authors (F.J.C.) thanks ESF Programme STOCHDYN for their financial support.

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